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LOCAL SELECTIONS AND LOCAL DENDRITES

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Let X be a Peano continuum, $C(X)$ its space of subcontinua, and $C(X, \varepsilon)$ the space of subcontinua of diameter less than ε. A selection on some subspace of $C(X)$ is a continuous choice function; the selection σ is rigid if $\sigma(A) \in B \subset A$ implies $\sigma(A) = \sigma(B)$. It is shown that X is a local dendrite (contains at most one simple closed curve) if and only if there exists $\varepsilon > 0$ such that $C(X, \varepsilon)$ admits a selection (rigid selection). Further, $C(X)$ admits a local selection at the subcontinuum A if and only if A has a neighborhood (relative to the space $C(X)$) which contains no cyclic local dendrite; moreover, that local selection may be chosen to be a constant.

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smooth dendroid	dendrite	local dendrite
simple closed curve	θ-curve	web
cyclic continuum		

1. Introduction

If X is a topological space, then 2^X denotes the space of nonempty closed subsets of X, endowed with the finite topology [9, 10]. If $\Sigma \subset 2^X$ then a *selection* for Σ is a continuous choice function $\sigma: \Sigma \to X$, i.e., $\sigma(A) \in A$ for each $A \in \Sigma$ and σ is continuous. For appropriate choices of Σ, the existence of a selection may reveal a good deal about the properties of X. For example, if X is a continuum, then 2^X admits a selection if and only if X is orderable. (This is due, essentially, to Michael [9]; see also Young [18]. A generalization to arbitrary compact Hausdorff spaces was established by Van Mill and Wattel [12].) A related theorem, due to Kuratowski, Nadler and Young [8], showed that subspaces of the real line are characterized by the existence of a selection for the family of subsets consisting of one or two points.

Characterizations of dendrites and other kinds of partially ordered continua, with a flavor similar to that of the results described above, have been established by considering selections for $C(X)$, the space of connected members of 2^X. It is a theorem of Nadler and the author [11] that $C(S^1)$ admits no selection. Further, we observed that the class of continua X for which $C(X)$ admits a selection lies properly between the smooth dendroids and the dendroids. As a consequence, if X is locally connected, then $C(X)$ admits a selection if and only if X is a dendrite. Subsequently the author [15] clarified the situation for dendroids as follows. Let us say that a choice function σ on a family of sets Σ (no topology assumed) is *rigid* provided, whenever A and B are members of Σ and $\sigma(A) \in B \subset A$, it follows that $\sigma(A) = \sigma(B)$. As we show in Theorem 2.1, a rigid choice function defines a tree-like partial order, and conversely. This partial order was exploited in [15] to show that a continuum X is a smooth dendroid if and only if $C(X)$ admits a rigid selection. In Theorem 3.2 we determine explicitly all of the rigid selections $C(X)$ admits.

It is natural to inquire whether a wider class of continua X can be characterized by the existence of a selection (or a rigid selection) on suitable subspaces of $C(X)$, and that inquiry is the main purpose of the present paper. Our principal results are theorems of this sort, mostly in the setting of Peano continua, i.e., continua which are metrizable and locally connected. In Theorem 4.2 it is shown that a Peano continuum is a local dendrite if and only if the space of subcontinua with sufficiently small diameter admits a selection. The corresponding result for rigid selections (Theorem 4.8) characterizes a much smaller class of continua. We also define a notion of local selection and we characterize those subcontinua of a Peano continuum at which a local selection exists.

The restriction to locally connected continua is essential to the arguments used here, especially the well-known fact that a Peano continuum which is not a dendrite must contain a simple closed curve. Since the related results of [11] and [15] are available for more general continua, it seems plausible that our hypotheses are unnecessarily restrictive and that generalizations to some kind of 'locally smooth' continua may be possible. But such an inquiry we leave to another paper.

Recall [1] that a Peano continuum admits a convex metric; therefore, whenever a metric is employed it is understood to be a convex metric. As a consequence, the ε-balls of a Peano continuum are arcwise connected.

2. Rigid choice functions

Rigid selections were studied tangentially by Franklin and Wallace [4] who noted that the 'least-element' map on a certain type of partially ordered compactum was an especially attractive selection. In [15] the author coined the term 'rigid' to describe this type of selection. Here we isolate the set-theoretic properties of rigid choice functions from the topological setting. (This is an exercise which was unstated but more or less implicit in [15].)

Recall that a *partial order* on a set X is a reflexive, antisymmetric, transitive relation $\Gamma \subset X \times X$. It is convenient to write $x \leq y$ as a synonym for $(x, y) \in \Gamma$, to write $x < y$ if $x \leq y$ and $x \neq y$, and to define

$$x\Gamma = \{y \in X : x \leq y\}, \qquad \Gamma x = \{y \in X : y \leq x\}.$$

If $A \subset X$ then the *Γ-zero* of A is an element $a_0 \in A$ such that $a_0 \leq a$ for all $a \in A$. The partial order Γ is of *tree type* if Γx is simply ordered, for each $x \in X$.

Theorem 2.1. *Let X be a set and suppose Σ is a family of nonempty subsets of X satisfying*
(1) *if $x \in X$ then $\{x\} \in \Sigma$,*
(2) *if $A, B \in \Sigma$ and $A \cap B \neq \emptyset$ then $A \cup B \in \Sigma$.*
If Σ admits a rigid choice function σ, then the relation

$$\Gamma_\sigma = \bigcup\{\{\sigma(A)\} \times A \colon A \in \Sigma\}$$

is a partial order of tree type on X, and $\sigma(A)$ is the Γ_σ-zero of A, for each $A \in \Sigma$.

Conversely, if Γ is a partial order on X such that each $A \in \Sigma$ has a Γ-zero, $\sigma(A)$, then σ is a rigid choice function on Σ and $\Gamma_\sigma \subset \Gamma$. Moreover, if $x\Gamma \in \Sigma$ for each $x \in X$ then $\Gamma_\sigma = \Gamma$.

Proof. Suppose σ is a rigid choice function for Σ; we show that Γ_σ has the required properties.

Γ_σ is reflexive. If $x \in X$ then $\{x\} \in \Sigma$ by (1) and hence $(x, x) = (\sigma(\{x\}), x) \in \Gamma_\sigma$.

Γ_σ is antisymmetric. If $x \leq y$ and $y \leq x$ then there are $A, B \in \Sigma$ such that $x = \sigma(A)$, $y \in A$ and $y = \sigma(B)$, $x \in B$. By (2), $(A \cup B) \in \Sigma$ and hence $\sigma(A \cup B) \in A$ or $\sigma(A \cup B) \in B$. If $\sigma(A \cup B) \in A$ then, by rigidity, $\sigma(A \cup B) = x$. Since $x \in B$ it follows by rigidity that $y = \sigma(B) = x$. A symmetric argument applies if $\sigma(A \cup B) \in B$.

Γ_σ is transitive. If $x \leq y$ and $y \leq z$ then there are $C, D \in \Sigma$ such that $x = \sigma(C)$, $y \in C$ and $y = \sigma(D)$, $z \in D$. By (2) we have $(C \cup D) \in \Sigma$. If $\sigma(C \cup D) \in C$ then, by rigidity, $x = \sigma(C \cup D)$ and hence $x \leq z$. If $\sigma(C \cup D) \in D$ then, by rigidity, $y = \sigma(C \cup D)$ and hence $y \leq x$. Since Γ_σ is antisymmetric we infer that $x = y$ and hence $x \leq z$.

Γ_σ is of tree type. If $x \leq z$ and $y \leq z$ then there are $E, F \in \Sigma$ such that $x = \sigma(E)$, $y = \sigma(F)$ and $z \in (E \cap F)$. By (2), $(E \cup F) \in \Sigma$. If $\sigma(E \cup F) \in E$ then, by rigidity, $x = \sigma(E \cup F)$ and hence $x \leq y$. If $\sigma(E \cup F) \in F$ then, by a symmetric argument, $y \leq x$.

It is obvious that $\sigma(A)$ is the Γ_σ-zero of A since $\{\sigma(A)\} \times A \subset \Gamma_\sigma$.

For the converse suppose Γ is a partial order on X such that each $A \in \Sigma$ has a Γ-zero, $\sigma(A)$. It is obvious that σ is a choice function. If $A, B \in \Sigma$ with $\sigma(A) \in B \subset A$ then $\sigma(A)$ is the Γ-zero of B and hence $\sigma(A) = \sigma(B)$, i.e. σ is rigid. It is immediate from the definition of Γ_σ that $\Gamma_\sigma \subset \Gamma$. Finally, if $x\Gamma \in \Sigma$ for each $x \in X$ and if $(x, y) \in \Gamma$ then $(x, y) \in \{\sigma(x\Gamma)\} \times (x\Gamma) \subset \Gamma_\sigma$. That is, $\Gamma_\sigma = \Gamma$. $\square$

3. Rigid selections, dendroids and dendrites

The following theorem was established by Nadler and the author [11].

Theorem 3.1. *$C(S^1)$ admits no selection.*

A *dendroid* is a continuum which is arcwise connected and hereditarily unicoherent. Dendroids have been characterized in terms of their intrinsic partial order structures [14], which are defined as follows. If X is a dendroid, fix a point $e \in X$ and define $x \leqslant_e y$ if and only if $x \in [e, y]$, the (unique) arc whose endpoints are e and y. It is well known that $\leqslant_e$ is a partial order of tree type, sometimes called the *weak-cutpoint partial order with basepoint* e [6]. A dendroid X is *smooth* at e if $e \in X$ and $\leqslant_e$ is a closed subset of $X \times X$. A dendroid is *smooth* if it is smooth at at least one of its elements. (Note that this is not the customary definition of smoothness, but it is an equivalent formulation; see for example [5] and [6].) In [15] it was shown that a continuum X is a smooth dendroid if and only if $C(X)$ admits a rigid selection. In order to keep the exposition reasonably self-contained we include a sketch of that argument. In addition we determine explicitly all of the rigid selections for $C(X)$.

For a dendroid X and $e \in X$, let $\leqslant_e$ denote the weak-cutpoint partial order with basepoint e and let $\Sigma = C(X)$. It is known [14] that each member of $C(X)$ has a zero relative to $\leqslant_e$ and hence the function of σ_e defined by

$$\sigma_e(A) = \text{zero of } A$$

is a rigid choice function.

Theorem 3.2. *Let X be a continuum. If X is a dendroid which is smooth at the point e then σ_e is a rigid selection for $C(X)$. Conversely, if σ is a rigid selection for $C(X)$ then X is a dendroid and $\sigma = \sigma_e$ where e is some point at which X is smooth.*

Proof. Suppose X is smooth at e; we show that σ_e is continuous. Let A_n be a sequence in $C(X)$ and suppose $\lim A_n = A \in C(X)$. If z is a subsequential limit of $\sigma_e(A_n)$ then $z \in A$ and hence $\sigma_e(A) \leqslant_e z$. On the other hand, there exists $a_n \in A_n$ such that $\lim a_n = \sigma_e(A)$; since $\sigma_e(A_n) \leqslant_e a_n$ for each n and since $\leqslant_e$ is closed, it follows that $z \leqslant_e \sigma_e(A)$. Therefore $z = \sigma_e(A)$ and we conclude that $\lim \sigma_e(A_n) = \sigma_e(A)$, i.e., σ_e is continuous.

Conversely, if X is a continuum and σ is a rigid selection for $C(X)$, it follows from [11] that X is a dendroid. We will show that Γ_σ is closed and that Γ_σ is the weak-cutpoint partial order with basepoint $m = \sigma(X)$. This will complete the proof.

To see that Γ_σ is closed, let (x_n, y_n) be a sequence in Γ_σ with $\lim(x_n, y_n) = (x, y)$. Then for each n there exists $A_n \in C(X)$ such that $x_n = \sigma(A_n)$ and $y_n \in A_n$. Since $C(X)$ is compact we may assume $\lim A_n = A \in C(X)$ and hence $y \in A$. Since σ is continuous it follows that $x = \sigma(A)$ and therefore $(x, y) \in \Gamma_\sigma$.

To see that Γ_σ coincides with $\leqslant_m$, suppose $x \leqslant_m y$. Then $x \in [m, y]$. Since $\{[t, y]: t \in [m, y]\}$ is an arc in $C(X)$ and since σ is continuous, the set

$$A_y = \{\sigma([t, y]): t \in [m, y]\}$$

contains the arc $[m, y]$. Therefore $\sigma([t, y]) = x$ for some $t \in [m, y]$ and hence $(x, y) \in \Gamma_\sigma$. On the other hand, if $(x, y) \in \Gamma_\sigma$ then there exists $A \in C(X)$ with $x = \sigma(A)$ and

$y \in A$. Since X is a dendroid, $[x, y] \subset A$ and so, by rigidity, $\sigma([x, y]) = x$. Again by the properties of dendroids, the arcs $[m, x]$ and $[m, y]$ meet in an arc $[m, t]$ with $t \in [x, y]$; by rigidity, $\sigma([t, x]) = x$ and $\sigma([m, x]) = m$. Applying the continuity of σ, it follows that $\sigma([q, t]) = t$ for some $q \in [m, t]$. By rigidity and because $[t, x] \subset [q, x]$ we infer that $t = \sigma([q, x]) = \sigma([t, x]) = x$ and hence $x \in [m, y]$; that is, $x \leqslant_m y$. □

A dendrite may be characterized as a locally connected—hence, smooth—dendroid, and so the following corollary may be deduced from Theorem 3.2.

Corollary 3.3. *If X is a Peano continuum then the following statements are equivalent*:
(1) *X is a dendrite,*
(2) *$C(X)$ admits a selection,*
(3) *$C(X)$ admits a rigid selection.*

Proof. It is obvious that (3) implies (2). To see that (2) implies (1) suppose σ is a selection for $C(X)$. Then $\sigma | C(K)$ is a selection for each subcontinuum $K \subset X$, and therefore, by Theorem 3.1, X contains no simple closed curve, i.e., X is a dendrite. That (1) implies (3) follows at once from Theorem 3.2. □

4. Local dendrites

If X is a continuum and $\varepsilon > 0$, we write $C(X, \varepsilon)$ to denote the subspace of $C(X)$ whose members have diameter less than ε. It is hardly a surprise that the situation concerning the existence of selections and rigid selections is quite different for $C(X, \varepsilon)$ than for $C(X)$. For Peano continua we are able to give complete answers to the existence questions.

A metric space is a *local dendrite* if each of its points has a closed neighborhood which is dendrite. It is well known [7, 13] that a compact metric space is a local dendrite if and only if X contains no small simple closed curves, i.e., if there is $\varepsilon > 0$ such that if K is a simple closed curve contained in X then $\operatorname{diam}(K) \geqslant \varepsilon$. Consequently, a compact local dendrite contains only finitely many simple closed curves.

The discussion which follows is facilitated by the following lemma on dendrites. Recall that if X is a dendrite and if a and b are distinct elements of X, then X contains exactly one arc whose endpoints are a and b. We denote that arc with the symbol $[a, b]$. Further, we define $[a, a] = \{a\}$ and $(a, b) = [a, b] - \{a, b\}$.

Lemma 4.1. *Let X be a dendrite, let a and b be elements of X and suppose a_n and b_n are sequences in X which converge to a and b, respectively. Then $[a_n, b_n]$ converges to $[a, b]$.*

Proof. Since X is a dendrite it follows from a result of Charatonik and Eberhart [3] that each point of (a, b) lies in $[a_n, b_n]$ for sufficiently large, n, and hence

$[a, b] \subset \liminf [a_n, b_n]$. At the same time, if $x \in X - [a, b]$ then there is an arcwise connected open set U such that

$$[a, b] \subset U \subset \bar{U} \subset X - \{x\}.$$

Therefore, for large n, $[a_n, b_n] \subset U$ and hence $x \in X - \limsup [a_n, b_n]$. It follows that $[a, b] = \lim [a_n, b_n]$.

Theorem 4.2. *If X is a Peano continuum then X is a local dendrite if and only if there exists $\varepsilon > 0$ such that $C(X, \varepsilon)$ admits a selection.*

Proof. If σ is a selection for $C(X, \varepsilon)$ and X is not a local dendrite, then X contains a simple closed curve K with $\operatorname{diam}(K) < \varepsilon$. But then $C(K) \subset C(X, \varepsilon)$ and so $\sigma | C(K)$ is a selection for $C(K)$, contrary to Theorem 3.1.

To prove the converse suppose X is a local dendrite. Since X contains at most finitely many simple closed curves, there is a finite set $\{x_1, \ldots, x_n\} \subset X$ such that $X - \{x_1, \ldots, x_n\}$ is connected and contains no simple closed curve. Let d be the (convex) metric on X, and for each $i = 1, \ldots, n$ and each $r > 0$ let

$$U_i(r) = \{y \in X \colon d(x_i, y) < r\}.$$

Choose $\varepsilon > 0$ so that the following are satisfied: if $A \in C(X, \varepsilon)$ then A is a dendrite; if $i \neq j$ then $\overline{U_i(4\varepsilon)}$ and $\overline{U_j(4\varepsilon)}$ are disjoint dendrites; and if

$$Y_k = X - \bigcup_{i=1}^{n} \{U_i(k\varepsilon)\}, \quad k = 1, 2, 3,$$

then Y_1, Y_2 and Y_3 are connected. It follows that Y_1, Y_2 and Y_3 are dendrites. Choose

$$x_0 \in X - \bigcup_{i=1}^{n} \{\overline{U_i(4\varepsilon)}\}$$

and let σ_0 be the rigid selection for $C(Y_1)$, with $\sigma_0(Y_1) = x_0$, which is guaranteed by Theorem 3.2. Similarly, for each $i = 1, \ldots, n$, let σ_i be the rigid selection for $C(\overline{U_i(4\varepsilon)})$ with $\sigma_i \overline{U_i(4\varepsilon)}) = x_i$.

We proceed to define a selection for $C(X, \varepsilon)$. Let $\sigma(A) = \sigma_0(A)$ in case $A \in C(X, \varepsilon)$ and $A \cap Y_3 \neq \emptyset$; let $\sigma(A) = \sigma_i(A)$ in case $A \in C(X, \varepsilon)$ and $A \subset U_i(2\varepsilon)$. Otherwise, if $A \in C(X, \varepsilon)$ then there is a unique i such that $A \subset U_i(3\varepsilon) \cap Y_1$, and hence $\sigma_0(A)$ and $\sigma_i(A)$ are each defined. Denote by $[\sigma_0(A), \sigma_i(A)]$ the unique arc in A which joins $\sigma_0(A)$ and $\sigma_i(A)$, and let $d_0(A) = d(\sigma_0(A), x_i)$ and $d_i(A) = d(\sigma_i(A), x_i)$. Note that d_0 and d_i are both continuous and that $0 \leq d_0(A) - d_i(A) < \varepsilon$. Let $\sigma(A)$ be the unique point of $[\sigma_0(A), \sigma_i(A)]$ such that

$$d(\sigma(A), x_i) = [d_0(A)(d_0(A) - d_i(A))/\varepsilon] + 3d_i(A) - 2d_0(A).$$

Thus σ has been defined for all $A \in C(X, \varepsilon)$ and it is obvious that σ is a choice function. To verify the continuity of σ at an element A of $C(X, \varepsilon)$, let A_n be a sequence in $C(X, \varepsilon)$ which converges to A. We consider three cases.

Case 1. $A \subset U_i(3\varepsilon) \cap Y_1$. In this case $A_n \subset U_i(3\varepsilon) \cap Y_1$ eventually, so we may assume $\sigma_0(A_n)$ and $\sigma_i(A_n)$ are defined. Because σ_0 and σ_i are continuous it follows that $\lim \sigma_0(A_n) = \sigma_0(A)$ and $\lim \sigma_i(A_n) = \sigma_i(A)$. By Lemma 4.1, the arcs $[\sigma_0(A_n), \sigma_i(A_n)]$ converge to $[\sigma_0(A), \sigma_i(A)]$ and hence all limit points of the sequence $\sigma(A_n)$ lie in $[\sigma_0(A), \sigma_i(A)]$. By the continuity of d_0 and d_i we see that

$$\lim d(\sigma(A_n), x_i) = d(\sigma(A), x_i),$$

and since d is convex we infer that $\lim \sigma(A_n) = \sigma(A)$.

Case 2. $A \cap Y_3 \neq \emptyset$. If $A_n \cap Y_3 \neq \emptyset$ eventually, then $\lim \sigma(A_n) = \lim \sigma_0(A_n) = \sigma_0(A) = \sigma(A)$. Hence we may assume $A_n \cap Y_3 = \emptyset$ on some subsequence. Both $\sigma_0(A_n)$ and $\sigma_i(A_n)$ are defined on this subsequence so, by Lemma 4.1, the arcs $[\sigma_0(A_n), \sigma_i(A_n)]$ converge to $[\sigma_0(A), \sigma_i(A)]$. Further, $d_0(A) = 3\varepsilon$ and hence

$$\lim d(\sigma(A_n), x_i) = \lim[3(3\varepsilon - d_i(A_n)) + 3d_i(A_n) - 6\varepsilon] = 3\varepsilon.$$

Since d is convex we see that $\lim \sigma(A_n) = \sigma(A)$.

Case 3. $A \subset \overline{U_i(2\varepsilon)}$. If $A_n \subset \overline{U_i(2\varepsilon)}$ eventually, then $\lim \sigma(A_n) = \lim \sigma_i(A_n) = \sigma_i(A) = \sigma(A)$, so we may assume $A_n - \overline{U_i(2\varepsilon)} \neq \emptyset$ on some subsequence. Therefore both $\sigma_0(A_n)$ and $\sigma_i(A_n)$ are defined on this subsequence, so by Lemma 4.1, the arcs $[\sigma_0(A_n), \sigma_i(A_n)]$ converge to $[\sigma_0(A), \sigma_i(A)]$. Further, $d_0(A) = 2\varepsilon$ and hence

$$\begin{aligned} \lim d(\sigma(A_n), x_i) &= \lim[2(2\varepsilon - d_i(A_n)) + 3d_i(A_n) - 4\varepsilon] \\ &= \lim d_i(A_n) = d_i(A). \end{aligned}$$

Since d is convex it follows that $\lim \sigma(A_n) = \sigma_i(A) = \sigma(A)$.

All cases have been examined, so the proof is complete. □

It is interesting to note, in the proof of Theorem 4.2, that the partial selections $\sigma_0, \sigma_1, \ldots, \sigma_n$ are all rigid but that, in general, σ is not rigid. The question arises as to which local dendrites X have the property that, for some $\varepsilon > 0$, the space $C(X, \varepsilon)$ admits a rigid selection. The answer, given in Theorem 4.8, is surprisingly restrictive.

To approach this problem we consider first the possible rigid selections for $C([0, 1])$. Theorem 3.2. tells us precisely what these selections are; they are the selections σ_t, where $t \in [0, 1]$. In terms of the natural ordering of $[0, 1]$ these are described as follows:

$$\sigma_0(A) = \min A, \qquad \sigma_1(A) = \max A \quad \text{for each } A \in C([0, 1]).$$

If $0 < t < 1$ then

$$\sigma_t(A) = \begin{cases} \max A & \text{if } A \subset [0, t], \\ t & \text{if } t \in A, \\ \min A & \text{if } A \subset [t, 1]. \end{cases}$$

From these remarks the following lemma is obvious.

Lemma 4.3. *If σ is a rigid selection on $C([0,1])$ and if $0<a<b<1$ then it is not possible to have $\sigma([0,a])=0$ and $\sigma([b,1])=1$.*

Lemma 4.4. *A rigid selection for $C([0,1],\varepsilon)$ has a unique extension to a rigid selection for $C([0,1])$.*

Proof. Let t_1 and t_2 be distinct elements of $[0,1]$, say $0\leq t_1<t_2\leq 1$. If $\delta=\min\{t_2-t_1,\frac{1}{2}\varepsilon\}$ then $[t_1,t_1+\delta]\in C([0,1],\varepsilon)$, $\sigma_{t_1}([t_1,t_1+\delta])=t_1$, and $\sigma_{t_2}([t_1,t_1+\delta])=t_1+\delta$. Thus any extension of a selection for $C([0,1]),\varepsilon)$ to a rigid selection for $C([0,1])$ is necessarily unique. It remains to show that if σ is a rigid selection for $C([0,1],\varepsilon)$ then $\sigma=\sigma_t\,|\,C([0,1],\varepsilon)$ for some $t\in[0,1]$. We divide the argument into two cases.

Case 1. *There is $[a,b]\in C([0,1],\varepsilon)$ such that $a<\sigma([a,b])<b$.* We claim, first, that if $[c,d]\in C([0,1],\varepsilon)$ with $c\leq d\leq\sigma([a,b])$ then $\sigma([c,d])=d$. By rigidity this is true for $[a,\sigma([a,b])]$. Since the rigid selections for $C([a,\sigma([a,b])$ are known, it follows that $\sigma([x,y])=y$ whenever $a\leq x\leq y\leq\sigma([a,b])$. By the continuity of σ there is a $\delta>0$ such that $\sigma([a-\delta,y])\in[a,y]$ if $a<y\leq\sigma([a,b])$ and hence, by rigidity, $\sigma([a-\delta,y])=\sigma([a,y])=y$. We may repeat this argument until all $[c,d]\subset C([0,\sigma([a,b])],\varepsilon)$ have been covered. By a symmetric argument it follows that if $[c,d]\in C[0,1],\varepsilon)$ with $\sigma([a,b])\leq c\leq d$ then $\sigma([c,d])=c$. Finally, if $[c,d]\in C[0,1],\varepsilon)$ with $c\leq\sigma([a,b])\leq d$ then a simple rigidity argument establishes that $\sigma([c,d])=\sigma([a,b])$. Letting $t=\sigma([a,b])$ it follows that $\sigma=\sigma_t\,|\,C([0,1],\varepsilon)$.

Case 2. *If $[a,b]\in C([0,1],\varepsilon)$ then $\sigma([a,b])=a$ or $\sigma([a,b])=b$.* Suppose $\sigma([0,\frac{1}{2}\varepsilon])=0$; by continuity $\sigma([0,\mu])=0$ for all $\mu<\varepsilon$. Since the rigid selections for $C([0,\varepsilon])$ are known, we see that $\sigma([\delta,\mu])=\delta$ for all $\delta\leq\mu<\varepsilon$. By repeated applications of the continuity of σ we extend this argument to all $[\delta,\mu]\subset[0,1]$ where $0<\mu-\delta<\varepsilon$, and hence $\sigma=\sigma_0\,|\,C([0,1],\varepsilon)$. If $\sigma([0,\frac{1}{2}\varepsilon])=\frac{1}{2}\varepsilon$ then a similar argument can be used to show that $\sigma([\delta,\mu])=\mu$ whenever $0<\mu-\delta<\varepsilon$, and hence $\sigma=\sigma_1\,|\,C([0,1],\varepsilon)$. □

Next we consider the rigid selections on $C(S^1,\varepsilon)$. To this end it is helpful to adopt the following conventions. The elements of S^1 are identified with the real numbers modulo 2π. An arc lying in S^1 is denoted $[a,b]$ where *a precedes b in the usual (positive) orientation of S^1*. (Thus, for example, $\pi\in[\frac{1}{2}\pi,\frac{3}{2}\pi]$ but $\pi\notin[\frac{3}{2}\pi,\frac{1}{2}\pi]$.) If $\varepsilon\leq 2$ then it is easy to see that $C(S^1,\varepsilon)$ admits two distinct rigid selections, σ^+ and σ^-, defined by

$$\sigma^+([a,b])=b,\qquad \sigma^-([a,b])=a.$$

In fact, these are the only rigid selections for $C(S^1,\varepsilon)$.

Theorem 4.5. *If $\varepsilon\leq 2$ and σ is a rigid selection on $C(S^1,\varepsilon)$ then $\sigma=\sigma^+$ or $\sigma=\sigma^-$.*

Proof. By Lemma 4.4, $\sigma\,|\,C([0,\pi],\varepsilon)$ has a unique extension to a rigid selection $\bar{\sigma}$ on $C([0,\pi])$, and therefore $\bar{\sigma}=\sigma_t$ for some t, $0\leq t\leq\pi$.

Suppose $0<t<\pi$; we choose δ so that $0<\delta<\min\{\frac{1}{2}\varepsilon, \frac{1}{2}t\}$. It follows that $\sigma([0, \delta]) = \bar{\sigma}([0, \delta]) = \delta$. But now we consider $\sigma|C([2\delta, \delta], \varepsilon)$; again by Lemma 4.4 this restriction extends uniquely to a rigid selection $\bar{\bar{\sigma}}$ on $C([2\delta, \delta])$. The selections $\bar{\sigma}$ and $\bar{\bar{\sigma}}$ must agree on $C([2\sigma, \pi])$; because t precedes 0 and δ in the ordering of $[2\delta, \delta]$ it follows that $\bar{\bar{\sigma}}([0, \delta]) = 0$. But $\bar{\bar{\sigma}}([0, \delta]) = \sigma([0, \delta]) = \bar{\sigma}([0, \delta]) = \delta$, and this is a contradiction.

Therefore $t=0$ or $t=\pi$. If $t=0$, so that $\bar{\sigma}=\sigma_0$ on $C([0, \pi])$, then $\bar{\sigma}([a, \pi]) = a$ for each a, $0\leq a\leq\pi$. By Lemma 4.4., $\sigma|C([\frac{1}{2}\pi, 0], \varepsilon)$ extends uniquely to a rigid selection σ' on $C([\frac{1}{2}\pi, 0])$, and $\bar{\sigma}$ and σ' agree on $C([\frac{1}{2}\pi, \pi])$. It follows that $\sigma' = \sigma_{\pi/2}$ on $C([\frac{1}{2}\pi, 0])$ and hence $\sigma'([a, b]) = a$ if $\frac{1}{2}\pi\leq a\leq b\leq 2\pi$. Hence $\sigma=\sigma^-$. If $t=\pi$ then, by a similar argument, $\sigma=\sigma^+$. □

Recall [17] that a *θ-curve* is a continuum representable as the union of two simple closed curves which meet in a (non-degenerate) arc. The following corollary is an easy consequence of Theorem 4.5.

Corollary 4.6. *If X is a θ-curve and $\varepsilon>0$ then $C(X, \varepsilon)$ admits no rigid selection.*

The following simple result on θ-curves is readily verifiable. It will be helpful in the proof of Theorem 4.8.

Lemma 4.7. *Let K_1 and K_2 be distinct simple closed curves contained in a Hausdorff space. If $K_1\cap K_2$ contains more than one point them $K_1\cup K_2$ contains a θ-curve.*

Theorem 4.8. *Let X be a Peano continuum. There exists $\varepsilon>0$ such that $C(x, \varepsilon)$ admits a rigid selection if and only if X contains at most one simple closed curve.*

Proof. If X contains no simple closed curve then X is a dendrite and the existence of the desired rigid selection follows from Corollary 3.3. If X contains a single simple closed curve K, then the closure of each component of $X-K$ is a dendrite which meets K in exactly one point. For each such component T_α let t_α be the point of $\bar{T}_\alpha\cap K$ and let σ_α denote the rigid selection on $C(\bar{T}_\alpha)$ such that $\sigma_\alpha(\bar{T}_\alpha) = t_\alpha$. Define σ on $C(X, \varepsilon)$ by $\sigma|C(\bar{T}_\alpha)\cap C(X, \varepsilon) = \sigma_\alpha$ for each α, and if $A\in C(X, \varepsilon) - \bigcup\{C(\bar{T}_\alpha)\}$ then

$$\sigma(A) = \sigma^+(A\cap K).$$

It is a straightforward exercise to see that σ is continuous and rigid.

Now suppose X contains two distinct simple closed curves, K_1 and K_2. We will show that $C(X, \varepsilon)$ admits no rigid selection. For if $C(X, \varepsilon)$ admits such a selection σ, then by Lemma 4.7 and Corollary 4.6 it follows that K_1 and K_2 meet in at most a point. Hence there is a unique (possibly degenerate) arc $[p_1, p_2]$ such that $[p_1, p_2]\cap K_1 = \{p_1\}$ and $[p_1, p_2]\cap K_2 = \{p_2\}$. By Theorem 4.5, $\sigma|C(K_i, \varepsilon)$ is of type σ^+ or σ^- for each $i=1, 2$. Hence there are points $a_1\in K_1-\{p_1\}$ and $a_2\in K_2-\{p_2\}$ such that

K_1 contains an arc $[p_1, a_1]$ which is a member of $C(X, \varepsilon)$, K_2 contains an arc $[p_2, a_2]$ which is a member of $C(X, \varepsilon)$, and $\sigma([p_1, a_1]) = a_1$ and $\sigma([p_2, a_2]) = a_2$. Denote by $[a_1, a_2]$ the arc $[a_1, p_1] \cup [p_1, p_2] \cup [p_2, a_2]$. By Lemma 4.4, $\sigma | (C([a_1, a_2]) \cap C(X, \varepsilon))$ admits an extension $\bar{\sigma}$, a rigid selection on $C([a_1, a_2])$. But $\bar{\sigma}([a_1, p_1]) = a_1$ and $\bar{\sigma}([p_2, a_2]) = a_2$, contrary to Lemma 4.3.

5. Local selections

If X is a continuum, $A \in C(X)$ and $\varepsilon > 0$, we define

$$C(X, A, \varepsilon) = \{B \in C(X): H(A, B) < \varepsilon\},$$

where H denotes the Hausdorff metric on $C(X)$. We say that $C(X)$ *admits a local selection at* A provided there is an $\varepsilon > 0$ such that $C(X, A, \varepsilon)$ admits a selection. It is interesting that among Peano continua the existence of a local selection at A is equivalent to the existence of a local selection at A which is not only rigid but constant. In Theorem 5.4 we characterize those subcontinua of a Peano continuum X at which $C(X)$ admits a local selection.

That characterization depends on the notion of a web. A *web* is a non-degenerate, cutpoint-free local dendrite. We also define an *n-od* to be union of n arcs which intersect precisely in a common endpoint. That common endpoint is called the *vertex* of the n-od. (In the special case where $n = 2$ it is convenient to allow any point of a 2-od to be the vertex.)

If x is an element of the metric space X then $U_\varepsilon(x)$ denotes the open ε-ball with center x.

Theorem 5.1. *If W is a web and $\varepsilon > 0$ then $C(W, W, \varepsilon)$ admits no selection.*

Proof. Suppose, on the contrary, that there is a web W and $\varepsilon > 0$ so that $C(W, W, \varepsilon)$ admits a selection σ. Clearly, by reducing ε slightly, we may assume that σ is defined on $\overline{C(W, W, \varepsilon)}$. Further, since W is a local dendrite, we may assume that the range of σ is a dendrite contained in W. Because W is cutpoint-free that dendrite must be an n-od, T (in fact, a 2-od in case $\sigma(W)$ is not a branch point of W). Thus $\sigma(W)$ is the vertex of T. Moreover, there exists δ, with $0 < \delta < \varepsilon$ so that for each $x \in W$ the set $W - U_\delta(x)$ is connected and hence is an element of $C(W, W, \varepsilon)$. Let $A_0 = W - U_\delta(\sigma(W))$; then $\sigma(A_0)$ lies on the arc $[\sigma(W), p]$ of T, where p is an endpoint of T. For each $t \in [\sigma(W), p]$ let $A_t = W - U_\delta(t)$; thus $A_0 = A_{\sigma(W)}$. As t moves from $\sigma(W)$ to p along $[\sigma(W), p]$, it follows from the continuity of σ and the fact that $\sigma(A_t) \in A_t$ that $\sigma(A_t) \in [t, p] \cap A_t$. But eventually $d(t, p) < \delta$ and hence $[t, p] \cap A_t = \emptyset$. This is a contradiction. $\square$

Corollary 5.2. *$C(S^1)$ admits a local selection everywhere except at S^1; moreover, that local selection may be taken to be a constant.*

Proof. In view of Theorem 5.1 it suffices to exhibit a constant selection on $C(S^1, A, \varepsilon)$ where A is an arc contained in S^1 and $\varepsilon > 0$. We may choose ε so small that S^1 is not a member of $C(S^1, A, \varepsilon)$ and so that A has arclength at least 2ε. If p is the midpoint of A then

$$p \in \bigcap C(S^1, A, \varepsilon)$$

and hence $\sigma \equiv p$ is the desired constant selection. □

Corollary 5.2. is quite illuminating as to the existence of local selections for arbitrary Peano continua X and elements $A \in C(X)$. That situation is summed up in our final theorem.

If X is a Peano continuum then $\mathscr{H}(X)$ denotes the set of webs contained in X. If A is a subcontinuum of X then we define

$$A_\delta = \{y \in X\colon d(a, y) \leq \delta \text{ for some } a \in A\}.$$

We note that A_δ is a Peano continuum.

Lemma 5.3. *If X is a Peano continuum and $A \in C(X)$ then $A \in \overline{\mathscr{H}(X)}$ if and only if there is $\varepsilon > 0$ such that if $0 < \delta < \varepsilon$ then A and A_δ have no common cutpoint.*

Proof. If $A \in \overline{\mathscr{H}(X)}$ then there is a sequence W_n of webs in X such that $\lim W_n = A$. If p is a cutpoint of both A and A_δ then eventually p cuts W_n, a contradiction.

Conversely, suppose ε exists as prescribed. Let $0 < \delta < \varepsilon$ and choose $a_1, \ldots, a_n \in A$ so that

$$A \subset \bigcup_{i=1}^{n} \{U_\delta(a_i)\} \subset A_\delta .$$

Since A_δ is a Peano continuum and since A and A_δ have no common cutpoint, it follows that A lies in a cyclic element of A_δ and hence [17, p. 79] each distinct pair of a_i $(i = 1, \ldots, n)$ lies in a simple closed curve contained in A_δ. Indeed, these simple closed curves may be chosen so that their union forms a web W_δ and $H(A, W_\delta) < \varepsilon$. Hence $A \in \overline{\mathscr{H}(X)}$. □

Theorem 5.4. *If X is a Peano continuum and $A \in C(X)$ then the following statements are equivalent*:

(1) *$C(X)$ admits a local selection at A,*

(2) *$C(X)$ admits a rigid local selection at A,*

(3) *$C(X)$ admits a constant local selection at A,*

(4) *there exists $\varepsilon > 0$ such that $\bigcap C(X, A, \varepsilon) \neq \emptyset$,*

(5) *$A \in C(X) - \overline{\mathscr{H}(X)}$.*

Proof. It is obvious that (4) implies (3), (3) implies (2), and (2) implies (1). To see that (1) implies (5), let σ be a selection for $C(X, A, \varepsilon)$. If $A \in \overline{\mathscr{H}(X)}$ then there exists a web $W \in C(X, A, \varepsilon)$ and hence $C(W, W, \delta) \subset C(X, A, \varepsilon)$ for some $\delta > 0$. But then $\sigma | C(W, W, \delta)$ is a selection for $C(W, W, \delta)$, contrary to Theorem 5.1.

It remains to prove that (5) implies (4). If $A \in C(X) - \overline{\mathscr{H}(X)}$ and if $\varepsilon > 0$ then, by Lemma 5.3, there exists δ, $0 < \delta < \varepsilon$, so that A and A_δ have a common cutpoint, p. Thus $A_\delta = M \cup N$ where M and N are non-degenerate subcontinua, $M \cap N = \{p\}$, and A meets M and N in non-degenerate subcontinua. Choose μ so that $0 < \mu < \delta$ and there are points $a \in A \cap M$ and $b \in A \cap N$ with $\mu < \min\{d(p, a), d(p, b)\}$. It follows that $p \in \bigcap C(X, A, \mu)$. □

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